MINISTRY OF EDUCATION AND TRAINING
HANOI NATIONAL UNIVERSITY OF EDUCATION

*

DUONG TRONG LUYEN

SOME CLASSES OF DEGENERATE NONLINEAR ELLIPTIC AND HYPERBOLIC EQUATIONS

Specialty: Integral and Differential Equations
Code: 62 46 01 03

SUMMARY OF THE PHD THESIS IN MATHEMATICS

Hanoi - 2017

This thesis has been completed at the Hanoi National University of Education

Supervisor: Prof. Dr. Sc. Nguyen Minh Tri, Institute of Mathematics

Referee 1: Prof. Dr. Sc. Vu Ngoc Phat, Institute of Mathematics, VAST

Referee 2: Assoc. Prof. Dr. Hoang Quoc Toan, VNU University of science

Referee 3: Assoc. Prof. Dr. Cung The Anh, Hanoi National University of Education

To be defended at the University level in front of Assessment Council at..... on.....

The thesis is publicly available at the National Library and the Library of Hanoi National University of Education.

INTRODUCTION

1. HISTORY AND SIGNIFICANCE OF THE PROBLEM

The theory of partial differential equations, which first studied in the works of J. D'Alembert (1717 - 1783), L. Euler (1707 - 1783), D. Bernoulli (1700 - 1782), J. Lagrange (1736 - 1813), P. Laplace (1749 - 1827), S. Poisson (1781 - 1840) and J. Fourier (1768 - 1830), has been known as a key tool to describe the mechanical and analytical model of physics. With Riemann's works in the middle of $19th$ century, the theory of partial differential equations became an essential tool of many branches of mathematics. The late $19th$ century, H. Poincaré pointed out the dialectical relationship between the theory of partial differential equations and other mathematical branches. In the $20th$ century, the theory of partial differential equations was highly developed based on functional analysis, especially since the theory of generalized functions was established by S. L. Sobolev and L. Schwartz.

The study of the general elliptic equations and systems has played an important role in the theory of differential equations. Together with the development of mathematics as well as science and technology, many problems relating to regularity of solutions of non elliptic equations and systems are investigated. There are classes of elliptic degenerate equations which have properties similar to elliptic equations.

Elliptic equations are researched in the publications of Friedman (1958), L. Hörmander (1966), H. Brezis and L. Nirenberg (1983), B. Helffer and J. Nourrigat (1985) (see also the references therein). Recently, many experts around the world, especially in universities and research institutes in the US, Japan, Russia, France, Italy have been interested in nonlinear degenerate elliptic equations, see D. S. Jerison and J. M. Lee

(1987), V. V. Grushin (1970, 1971), T. Bieske (2002), A. E. Kogoj, E. Lanconelli and S. Sonner (2000, 2012 - 2016), C. J. Xu (1992) and the references therein. A number of Vietnamese authors have also obtained deep results for degenerate elliptic differential equations and systems as well as related equations, see N. M. Chuong, N. M. Tri, P. T. Thuy, N. T. C. Thuy, C. T. Anh, T. D. Ke (see also the references therein). They proved smoothness, analyticity, Gevrey regularity of solutions for Gilioli - Treves semilinear equations, equations the principle symbol of which are powers of the Mizohata operator, Kohn - Laplacian semilinear equations on the Heisenberg group, semilinear equations of the Grushin type and results on the existence and non-existence of solutions of boundary value problems for degenerate elliptic equations, results on the existence and asymptotic behaviour of the solution for degenerate parabolic equations, the result on the existence and asymptotic behaviour of the solution for degenerate hyperbolic equations. However the results for elliptic and hyperbolic equations are very sparse and for from complete. To fill this gap, we chose the research topic for my thesis "**Some classes of degenerate nonlinear elliptic and hyperbolic equations**".

2. GOALS OF THE THESIS

• *Topic 1*: Study elliptic boundary value problems containing degenerate elliptic operator Δ_γ with the following aspects:

- The existence of weak solutions of the problem;

- Regularity of weak solutions.

• *Topic 2*: Study damped hyperbolic equations containing strongly degenerate elliptic operators in bounded domains with the following aspects:

- The existence and uniqueness of mild solutions;

- The existence of a global attractor;

- Evaluation of the fractal dimension of the attractor.

• *Topic 3*: Study damped hyperbolic equations containing the Grushin operators in the whole space in the following aspects:

- The existence and uniqueness of mild solutions;

- The existence of a global attractor.

3. OBJECT AND SCOPE OF THE THESIS

The subjects of the thesis are the boundary value problems and the initial boundary value problems for equations containing degenerate elliptic operators

$$\Delta_\gamma = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\gamma_j^2 \frac{\partial}{\partial x_j}\right).$$

4. RESEARCH METHODS OF THE THESIS

• The existence of weak solution for the problem is studied by using variational methods in the theory of real functions.

• The regularity of solution is investigated by using Sobolev-type embedding theorems and interpolation.

• The existence and uniqueness of solutions are studied by using fixed point theorems.

• The asymptotic behaviour of solutions are examined by using the methods from the theory of infinite dimensional dissipative dynamical systems.

• The fractal dimension of global attractors is evaluated by using the method of $\ell-$trajectories.

5. RESULTS OF THE THESIS

In the thesis we obtain the following main results

- For semilinear degenerate elliptic boundary value problem, we prove the existence of weak solution of the problem under some conditions on nonlinear terms. We also proved the regularity of weak solutions of the problems. This is the content of Chapter 2.

- For damped hyperbolic equations containing strongly degenerate elliptic operators in bounded domain, we prove the existence and uniqueness of mild solutions, the existence of the global attractor and evaluate the fractal dimension of the set of attractors. This is the content of Chapter 3.

- For damped hyperbolic equations containing the Grushin operator in the whole space, we establish the existence and uniqueness of mild solutions, the existence of global attractor. This is the content of Chapter 4.

The results of the thesis are novel, have scientific meaning, and contribute to the study of the existence of weak solution, smooth properties of solutions of semilinear degenerate elliptic boundary value problem, and asymptotic behaviour of the solutions of the damped hyperbolic equations containing degenerate elliptic operators.

6. STRUCTURE OF THE THESIS

Together with the Introduction, Overview, Conclusion, Author's works related to the thesis that have been published and References, the thesis includes four chapters:

- Chapter 1: *Preliminaries.* In this chapter, we present some preliminaries including: The $\Delta_\gamma-$ Laplace operator; some functional spaces and their important properties used in the thesis, the general results of the theory of a global attractor and some auxiliary results used for the next chapters.

• Chapter 2: *The existence and regularity of weak solution of boundary value problem for semilinear degenerate elliptic equations.* In this chapter, we study a class of elliptic boundary value problems containing the operator Δ_γ under some conditions on the nonlinear terms. We obtain results on the existence of nontrivial weak solutions, regularity of weak solutions.

• Chapter 3: *Global attractor for the damped hyperbolic equations containing strongly degenerate elliptic operators in bounded domains.* In this chapter, we study a class of damped hyperbolic equations containing strongly degenerate elliptic operators in bounded domains $\Omega \subset \mathbb{R}^N$ with the nonlinear terms that are of polynomial growth. We prove the existence of mild solutions of the problem, the existence of a global attractor in the space $S^2_{(k_1,k_2),0}(\Omega) \times L^2(\Omega)$ and we prove the finite fractal dimensionality of a global attractor.

• Chapter 4: *Global attractor for the damped hyperbolic equations containing the Grushin operators in* $\mathbb{R}^N$. In this chapter, we prove the existence of a global attractor for the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator with a locally Lipschitz on nonlinearity which satisfies a subcritical growth condition.

Chapter 1
PRELIMINARIES

In this chapter, we present some preliminaries including: The $\Delta_\gamma-$ Laplace operator; some functional spaces; important properties used in the thesis, the general results of the theory of a global attractor and some auxiliary results used in the next chapters.

1.1 The $\Delta_\gamma-$ Laplace operators and some functional spaces

1.1.1 The $\Delta_\gamma-$ Laplace operator

We assume Ω is a bounded domain with smooth boundary in $\mathbb{R}^N, N \geq 2$. We consider the operators of the form:

$$\Delta_\gamma = \sum_{j=1}^{N} \partial_{x_j} \left(\gamma_j^2 \partial_{x_j}\right), \ \partial_{x_j} = \frac{\partial}{\partial x_j}, j = 1, 2, \ldots, N,$$

here, the functions $\gamma_j : \mathbb{R}^N \to \mathbb{R}$ are assumed to be continuous, and $\gamma_j > 0$, $j = 1, 2, \ldots, N$ in $\mathbb{R}^N \backslash \Pi$, where

$$\Pi := \left\{ X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : \prod_{j=1}^{N} x_j = 0 \right\}.$$

Moreover, we make the following assumptions

i) $\gamma_1(X) \equiv 1, \gamma_j(X) = \gamma_j(x_1, x_2, \ldots, x_{j-1}), \ j = 2, \ldots, N;$

ii) For every $X \in \mathbb{R}^N$ we have $\gamma_j(X) = \gamma_j(X^*), \ j = 1, 2, \ldots, N,$ where

$$X^* = (|x_1|, \ldots, |x_N|) \text{ if } X = (x_1, x_2, \ldots, x_N);$$

iii) There exists a constant $\rho \geq 0$ such that :

$$0 \leq x_k \partial_{x_k} \gamma_j(X) \leq \rho \gamma_j(X), \forall k \in \{1, 2, \ldots, j-1\}, \forall j = 2, \ldots, N,$$

for every $X \in \overline{\mathbb{R}}_+^N := \left\{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_j \geq 0, \forall j = 1, 2, \ldots, N\right\}$;

iv) There exists a semigroup of dilations $\{\delta_t\}_{t>0}$ such that :

$$\begin{aligned} \delta_t : \mathbb{R}^N &\longrightarrow \mathbb{R} \\ (x_1, \ldots, x_N) &\longmapsto \delta_t(x_1, \ldots, x_N) = (t^{\varepsilon_1} x_1, \ldots, t^{\varepsilon_N} x_N) \end{aligned}$$

where $1 = \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_N$, such that γ_j is δ_t - homogeneous of degree $\varepsilon_j - 1$, i. e.,

$$\gamma_j(\delta_t(X)) = t^{\varepsilon_j - 1} \gamma_j(X), \forall X \in \mathbb{R}^N, \forall t > 0, \ j = 1, \ldots, N.$$

The number

$$\widetilde{N} := \sum_{j=1}^{N} \varepsilon_j$$

is called the homogeneous dimension of $\mathbb{R}^N$ with respect to $\{\delta_t\}_{t>0}$.

1.1.2 Some functional spaces

Definition 1.1.1. By $S_\gamma^p(\Omega)$ $(1 \leq p < +\infty)$ we will denote the set of all functions $u \in L^p(\Omega)$ such that $\gamma_j \partial_{x_j} u \in L^p(\Omega)$ for all $j = 1, \ldots, N$. We define a norm in this space as follows

$$\|u\|_{S_\gamma^p(\Omega)} = \left\{ \int_\Omega \left(|u|^p + \sum_{j=1}^{N} \left|\gamma_j \partial_{x_j} u\right|^p \right) \mathrm{d}x \right\}^{\frac{1}{p}}.$$

If $p = 2$ we can also define the scalar product in $S_\gamma^2(\Omega)$ as follows

$$(u, v)_{S_\gamma^2(\Omega)} = (u, v)_{L^2(\Omega)} + \sum_{j=1}^{N} \left(\gamma_j \partial_{x_j} u, \gamma_j \partial_{x_j} v\right)_{L^2(\Omega)}.$$

The space $S_{\gamma,0}^p(\Omega)$ is defined as the closure of $C_0^1(\Omega)$ in the normed space $S_\gamma^p(\Omega)$.

One can prove that $S_\gamma^p(\Omega)$ and $S_{\gamma,0}^p(\Omega)$ are Banach spaces, $S_\gamma^2(\Omega)$ and $S_{\gamma,0}^2(\Omega)$ are Hilbert spaces.

Put

$$\nabla_\gamma u := (\gamma_1 \partial_{x_1} u, \gamma_2 \partial_{x_2} u, \ldots, \gamma_N \partial_{x_N} u)\,, |\nabla_\gamma u| := \Big(\sum_{j=1}^{N} \left|\gamma_j \partial_{x_j} u\right|^2 \Big)^{\frac{1}{2}}.$$

If $\gamma = (\underbrace{1, 1, \ldots, 1}_{N_1-\text{ times}}, \underbrace{|x|^k, \ldots, |x|^k}_{N_2-\text{ times}})$, we denote $S^2_{k,0}(\Omega) := S^2_{\gamma,0}(\Omega)$, $\nabla_k := \nabla_\gamma$, and $N_k := \widetilde{N} = N_1 + (1+k)N_2$.

If $\gamma = (\underbrace{1, 1, \ldots, 1}_{(N_1+N_2)-\text{ times}}, \underbrace{|x|^{k_1}|y|^{k_2}, \ldots, |x|^{k_1}|y|^{k_2}}_{N_3-\text{ times}})$, we denote $S^2_{(k_1,k_2),0}(\Omega) := S^2_{\gamma,0}(\Omega)$, $\nabla_{k_1,k_2} := \nabla_\gamma$, and $N_{k_1,k_2} := \widetilde{N} = N_1 + N_2 + (1 + k_1 + k_2)N_3$.

Definition 1.1.2. We define the space $S^2_1(\mathbb{R}^N)$ as the completion of $C^\infty_0(\mathbb{R}^N)$ in the norm

$$\|u\|_{S^2_k(\mathbb{R}^N)} = \left\{ \int_{\mathbb{R}^N} \Big(|u|^2 + |\nabla_k u|^2 \Big)\, \mathrm{d}X \right\}^{\frac{1}{2}}.$$

Then $S^2_1(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$(u, v)_{S^2_k(\mathbb{R}^N)} = (u, v)_{L^2(\mathbb{R}^N)} + (\nabla_k u, \nabla_k v)_{L^2(\mathbb{R}^N)}.$$

1.1.3 Some important properties

In this subsection, we recall some important properties which we will use in our thesis such as: embedding Sobolev theorem, theorem on conditions for a functional to be bounded from below and to attain infimum.

1.2 Global attractor and properties

1.2.1 Some definitions

In this subsection, we recall some basic definitions semigroups, global attractors, gradient systems, asymptotically compact semigroup, orbit of

sets, fractal dimensions, generalized squeezing property.

1.2.2 Some properties

In this subsection, we recall some important properties which we will use in our thesis as: Aubin - Lions compact Lemma; Stone's theorem, theorems on the fractal dimension of a finite set.

Chapter 2

THE EXISTENCE AND REGULARITY OF WEAK SOLUTIONS OF BOUNDARY VALUE PROBLEM FOR SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS

In this chapter, we study a class of elliptic boundary value problems containing the operator Δ_γ under some conditions on the nonlinear terms and we obtain some results on the existence of nontrivial weak solutions, regularity of weak solutions. This chapter consists of two parts:

- Part one: Theorems on the existence of weak solutions.

- Part two: Regularity of solution to boundary value problems.

The contents of this chapter are based on the papers of Luyen and Tri [1], [2].

2.1 Some existence theorems weak solutions

In this section, we will study the existence of weak solutions of elliptic boundary value problem containing the degenerate elliptic operator Δ_γ, the method we use here is the variational method.

2.1.1 Theorem on existence of nontrivial weak solutions

In this section we consider the following problem

$$\begin{cases} \Delta_\gamma u + f(X,u) = 0 \ \text{ in } \ \Omega, \\ \qquad\qquad\qquad u = u_0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$, and $u_0 = u_0(X) \in C^2(\overline{\Omega})$. Using the sub-super solution and variational methods, we get the well-posedness of the problem.

Theorem 2.1.1. *Take $\widetilde{N} > 2$. Suppose that $\underline{u}(X) \in S_\gamma^2(\Omega)$ is a weak sub-solution while $\overline{u}(X) \in S_\gamma^2(\Omega)$ a weak super-solution to problem* (2.1) *and with constants $\underline{c}, \overline{c} \in \mathbb{R}$, it holds that $-\infty < \underline{c} \leq \underline{u}(X) \leq \overline{u}(X) \leq \overline{c} < +\infty$, almost everywhere in Ω. Furthermore, suppose that functions $f(x,\xi)$ satisfies $|f(X,\xi)| \leq a(X) + b|\xi|^p$ where*

$$a(X) \in L^{\frac{p}{p-1}}(\Omega), 1 < p < \frac{\widetilde{N}+2}{\widetilde{N}-2}, b \in \overline{\mathbb{R}}_+.$$

Then there exists a weak solution $u(X) \in S_\gamma^2(\Omega)$ of (2.1)*, satisfying the condition $\underline{u}(X) \leq u(X) \leq \overline{u}(X)$ almost everywhere in Ω.*

Corollary 2.1.2. *Let the conditions of Theorem 2.1.1 be satisfied. Then there exists a weak solution of the problem* (2.1) *in $L^\infty(\Omega)$.*

Corollary 2.1.3. *Suppose that $f(X,\xi) = k(X)\xi|\xi|^{q-2} - \xi|\xi|^{p-2}$, where $1 < k(X) \leq C < \infty$, C is a constant and $2 \leq q < p < \frac{2\widetilde{N}}{\widetilde{N}-2}$. Then the problem*

$$\begin{aligned} \Delta_\gamma u + f(X,u) = 0 \quad & in \;\; \Omega, \\ u = u_0 \quad & on \; \partial\Omega, \end{aligned}$$

where u_0 is a constant and $u_0 \geq 1$, has a weak solution in $L^\infty(\Omega)$.

2.1.2 Theorems on the existence of nonnegative weak solutions

In this section we consider the following problem

$$\begin{cases} -\Delta_\gamma u + \lambda u = u|u|^{p-2} & \text{in } \; \Omega, \\ u \geq 0 & \text{in } \; \Omega, \\ u = 0 & \text{on } \; \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$, and λ, p are constants.

Theorem 2.1.4. *Let $\widetilde{N} > 2, p \in \left(2, \frac{2\widetilde{N}}{\widetilde{N}-2}\right)$ and $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots,$ be the eigenvalues of the Dirichlet boundary value problem for the $(-\Delta_\gamma)-$*

Laplace operator in $S^2_{\gamma,0}(\Omega)$. *Then, for any* $\lambda > -\lambda_1$, *the problem* (2.2) *always has a nontrivial weak solution.*

2.2 Regularity of solutions to boundary value problems for degenerate elliptic equations

In this section we consider the following problem

$$\begin{cases} \Delta_\gamma u + f(X,u) = 0 \text{ in } \Omega, \\ \qquad\qquad\qquad u = 0 \text{ on } \ \partial\Omega, \end{cases} \tag{2.3}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$

The main result in this section is the following theorem:

Theorem 2.2.1. *Let* Ω *be a bounded domain with a smooth boundary in* $\mathbb{R}^N$. *Assume that* $f(X,\xi) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function with respect to* ξ *for almost all* $X \in \Omega$ *and measurable in* X *for all* $\xi \in \mathbb{R}$ *such that for almost every* $X \in \Omega$

$$|f(X,\xi)| \leq a(X)\,(1 + |\xi|)$$

with a function $a(X) \in L^{\frac{\widetilde{N}}{2}}_{loc}(\Omega)$. *Also let* $u \in S^2_{\gamma,loc}(\Omega)$ *be a weak solution of equation* (2.3). *Then* $u \in L^q_{loc}(\Omega)$ *for any* $q < \infty$.

Moreover, if $u \in S^2_{\gamma,0}(\Omega)$ *and* $a(X) \in L^{\frac{\widetilde{N}}{2}}(\Omega)$ *then* $u \in L^q(\Omega)$ *for any* $q < \infty$.

Theorem 2.2.2. *Let* $f(X,\xi) = f(\xi)$ *satisfy the following conditions*

1) $f(\xi) \in C^{0,\alpha}_{loc}(\mathbb{R})$ *where* $\alpha \in (0,1]$.

2) $|f(\xi)| \leq C(1+|\xi|^m)$ *for* C *is positive constants and* $1 < m < \frac{\widetilde{N}+2}{\widetilde{N}-2}$.

Then every weak solution of the problem (2.3) *belongs to* $L^p(\Omega)$ *for any* $1 \leq p < \infty$.

From Theorem 2.2.1 and Theorem 9 in N. M. Tri (1998), we have

Corollary 2.2.3. *Let $f(\xi)$ satisfy the conditions of Theorem 2.2.1 and in addition*

1) $f(\xi) = o(\xi)$ *when* $\xi \to 0$.

2) *The exists* A *and* $\mu \in [0, \frac{1}{2})$ *such that* $F(\xi) \leq \mu f(\xi)\xi$ *for all* $|\xi| \geq A$, *where* $F(\xi) = \int\limits_0^\xi f(x)\mathrm{d}x$.

Then there exists a weak solution of the problem (2.3) *which belongs to* $L^p(\Omega)$ *for any* $1 \leq p < \infty$.

Chapter 3

GLOBAL ATTRACTOR FOR THE DAMPED HYPERBOLIC EQUATIONS CONTAINING STRONGLY DEGENERATE ELLIPTIC OPERATORS IN BOUNDED DOMAINS

In this chapter, we study a class of damped hyperbolic equations containing strongly degenerate elliptic operators in bounded domains $\Omega \subset \mathbb{R}^N$ with the nonlinear terms having polynomial growth. We prove the existence of mild solutions of the problem, the existence of a global attractor in the space $S^2_{(k_1,k_2),0}(\Omega) \times L^2(\Omega)$ and we prove the finite fractal dimensionality of the global attractor. This chapter consists of three parts:

- Part one: The existence and uniqueness of mild solutions.
- Part two: The existence of a global attractor in the space $S^2_{(k_1,k_2),0}(\Omega) \times L^2(\Omega)$.
- Part three: The finiteness of the fractal dimension of the global attractors.

The contents of this chapter are based on the paper of Luyen and Tri [3].

3.1 Uniqueness of mild solutions

3.1.1 The setting of the problem

In this chapter we study the following problem

$$\begin{cases} u_{tt} + \beta u_t = P_{k_1,k_2} u + f(X,u), \quad X \in \Omega, t > 0, \\ u(X,t) = 0, \quad X \in \partial\Omega, t > 0, \\ u(X,0) = u_0(X), \quad u_t(X,0) = u_1(X), \end{cases} \tag{3.1}$$

where Ω is a bounded domain with smooth boundary in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3} = \mathbb{R}^N (N_1, N_2, N_3 \geq 1)$, β is a positive constant, $X = (x, y, z)$,

$$u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, |x|^{2k_1} = \left(\sum_{i=1}^{N_1} x_i^2 \right)^{k_1}, |y|^{2k_2} = \left(\sum_{j=1}^{N_2} y_j^2 \right)^{k_2}.$$

We assume that $f(X, \xi) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function with respect to ξ for almost all $X \in \Omega$ and measurable in X for all $\xi \in \mathbb{R}$ and satisfies:

$$|f(X, \xi_1) - f(X, \xi_2)| \leq C|\xi_1 - \xi_2| \Big(g(X) + |\xi_1|^\rho + |\xi_2|^\rho \Big), \tag{3.2}$$

with $0 \leq \rho \leq \frac{2}{N_{k_1,k_2} - 2}$, $N_{k_1,k_2} = N_1 + N_2 + (1 + k_1 + k_2)N_3 > 2$,

$$f(X, 0) = h(X) \in L^2(\Omega), \tag{3.3}$$

$$F(X, \xi) = \int_0^\xi f(X, \tau) d\tau \leq g_1(X) + g_2(X)\xi^2, \tag{3.4}$$

$$f(X, \xi)\xi \leq g_3(X) + g_4(X)\xi^2, \tag{3.5}$$

where $\rho, C > 0$ are constants; the functions

$$g_1(X), g_3(X) \in L^1(\Omega),\ g_2(X) \in L^{\frac{N_{k_1,k_2}}{2}}(\Omega),\ g_4(X) \in L^{\frac{N_{k_1,k_2}}{2}}(\Omega),$$
$$\|g_2\|_{L^{\frac{N_{k_1,k_2}}{2}}(\Omega)} < \frac{1}{2C^2(2^*_{k_1,k_2}, \Omega)},\ \|g_4\|_{L^{\frac{N_{k_1,k_2}}{2}}(\Omega)} < \frac{1}{C^2(2^*_{k_1,k_2}, \Omega)},$$

where $2^*_{k_1,k_2} = \dfrac{2N_{k_1,k_2}}{N_{k_1,k_2} - 2}$, $C(2^*_{k_1,k_2}, \Omega)$ is the best constant in the Sobolev inequality

$$\|u\|_{L^{2^*_{k_1,k_2}}(\Omega)} \leq C(2^*_{k_1,k_2}, \Omega) \|u\|_{S^2_{(k_1,k_2),0}(\Omega)},$$

$g(X)$ is a nonnegative function, $g(X) \in L^{N_{k_1,k_2}}(\Omega)$ if $\rho = 0$, and $g(X) \in L^{\frac{2^*_{k_1,k_2}}{\rho}}(\Omega)$ if $\rho \neq 0$.

We put

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, A = \begin{pmatrix} 0 & Id \\ P_{k_1,k_2} & 0 \end{pmatrix},$$

$$f^*(U)(X) = \begin{pmatrix} 0 \\ -\beta v(X) + f(X, u(X)) \end{pmatrix}, U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

The problems (3.1) is the equivalent to the problem

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = AU + f^*(U), \ t > 0, \\ U(0) = U_0. \end{cases} \tag{3.6}$$

We set $H = S^2_{(k_1,k_2),0}(\Omega) \times L^2(\Omega)$. We regard H as a Hilbert space with the inner product

$$(U, \overline{U})_H = (\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}) = (\nabla_{k_1,k_2} u, \nabla_{k_1,k_2} \overline{u})_{L^2(\Omega)} + (v, \overline{v})_{L^2(\Omega)}.$$

3.1.2 Uniqueness of mild solutions

Definition 3.1.1. Let $T > 0, T \in \mathbb{R}$. A (strongly) continuous mapping $U : [0, T) \to H$ is said to be a mild solution of the problem (3.1) if it solves the following integral equation

$$U(t) = e^{At} U_0 + \int_0^t e^{A(t-s)} f^*(U(s)) \mathrm{d}s, \quad t \in [0, T).$$

If U is (strongly) differentiable almost everywhere in $[0, T)$ with U_t and AU in $L^1_{loc}([0, T), H)$, U almost everywhere in $[0, T)$ satisfies the differential equation

$$\frac{dU}{dt} = AU + f^*(U) \quad \text{on } (0, T), \text{ and } U(0) = U_0, U(t) \in D(A),$$

then U is called a strong solution of the problem (3.1).

Theorem 3.1.2. *Assume that $f(X, u)$ satisfies the conditions* (3.2)-(3.4) *and $U_0 \in H$. Then the problem* (3.1) *has a unique global solution $U \in C([0, \infty); H)$. Moreover, for each fixed t the map $U_0 \to S(t)U_0 := U(t)$ is continuous on H.*

3.2 Existence of a global attractor in $S^2_{(k_1,k_2),0}(\Omega) \times L^2(\Omega)$

In view of Theorem 3.1.2, we can define a continuous semigroup $S(t) : H \to H$ as follows

$$S(t)U_0 := U(t),$$

where $U(t)$ is the unique global mild solution of (3.1) subject to U_0 as the initial datum.

Using the energy equation given by J. M. Ball in 2004 we prove the following theorem:

Theorem 3.2.1. *Assume that $f(X,u)$ satisfies the conditions* (3.2)-(3.5). *Then the semigroup $S(t)$ generated by* (3.1) *has a compact connected global attractor $\mathcal{A} = W^u(E)$ in H.*

Theorem 3.2.2. *Assume that $f(X,u)$ satisfies the conditions* (3.2)-(3.5). *Then the semigroup $S(t)$, defined by the problem* (3.1), *possesses a minimal compact global attractor $\mathcal{M}$ in the space $S^2_{1,0}(\Omega) \times L^2(\Omega)$. Moreover, for each $(u_0, u_1) \in S^2_{1,0}(\Omega) \times L^2(\Omega)$ the corresponding solution $(u(t), u_t(t)) = S(t)(u_0, u_1)$ tends to the set E of equilibrium points in $S^2_{1,0}(\Omega) \times L^2(\Omega)$ as $t \to +\infty$.*

3.3 Finite dimensionality of attractors

In this section we prove the fractal dimension of the global attractors $\mathcal{A}$ of the problem (3.1) is finite. The method we use here is the method of $\ell-$trajectories given by D. Pražák in 2002 and the results we achieve are:

Theorem 3.3.1. *Assume that $f(X,u)$ satisfies the conditions* (3.2)-(3.5) *with $0 < \rho < \dfrac{2}{N_{\alpha,\beta} - 2}$. Then the fractal dimension of the global attractor $\mathcal{A}$ is finite.*

Remark 3.3.2. If $\rho = 0$, $g(X) \in L^{N_{\alpha,\beta}+\epsilon}(\Omega)$, for every positive small ϵ then Theorem 3.3.1 holds.

Chapter 4

GLOBAL ATTRACTOR FOR THE DAMPED HYPERBOLIC EQUATIONS CONTAINING THE GRUSHIN OPERATORS IN $\mathbb{R}^N$

In Chapter 3, we have studied the problem of damped hyperbolic equation containing strongly degenerate elliptic operators in a bounded domain $\Omega \subset \mathbb{R}^N$. Then semigroup $S(t), t > 0$, generated by the problem is (nonlinear) compact semigroup, i.e., $S(t)$ is compact operators for every $t > 0$ (hence from embedding $S^2_{(k_1,k_2),0}(\Omega) \hookrightarrow L^2(\Omega)$ is compact). In this chapter, we continue to study this problems in whole space $\mathbb{R}^N, N \geq 2$. In this case, the necessary embedding are no longer compact, so $S(t)$ is no longer compact semigroup and it causes great difficulties in research. To overcome, we introduce a corresponding degenerate distance and use the method of estimating tail solution to prove the existence of a global attractors of semigroup generated by the problem in the space $S^2_k(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. This chapter consists of two parts:

- Part one: The existence and uniqueness of mild solutions.
- Part two: The existence of a global attractors in the spaces $S^2_k(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

The contents of this chapter are based on the paper of Luyen and Tri [4].

4.1 Uniqueness of mild solutions

4.1.1 The setting of the problem

In this chapter we study the following problem

$$\begin{cases} u_{tt} + \beta u_t + u = G_k u + f(X,u), t > 0, \\ \qquad\qquad X = (x,y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} := \mathbb{R}^N, \\ \qquad u(X,0) = u_0(X), u_t(X,0) = u_1(X), \end{cases} \tag{4.1}$$

where β is a positive constant, $u_0(X) \in S_k^2(\mathbb{R}^N), u_1(X) \in L^2(\mathbb{R}^N)$ and

$$u_t = \frac{\partial u}{\partial t}, u_{tt} = \frac{\partial^2 u}{\partial t^2}, |x|^{2k} = \left(\sum_{i=1}^{N_1} x_i^2\right)^k, G_k u = \Delta_x u + |x|^{2k}\Delta_y u.$$

We assume that $f(X,\xi) : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function satisfying

$$|f(X,\xi_1) - f(X,\xi_2)| \leq C_1|\xi_1 - \xi_2| \left(g(X) + |\xi_1|^\rho + |\xi_2|^\rho\right) \tag{4.2}$$

$$\text{with } \ 0 \leq \rho \leq \frac{2}{N_k - 2}, N_k = N_1 + (1+k)N_2 > 2,$$

$$f(X,0) = h(X) \in L^2(\mathbb{R}^N), \tag{4.3}$$

$$F(X,\xi) \geq C_2 f(X,\xi)\xi + g_1(X), \qquad \text{for all } X \in \mathbb{R}^N, \xi \in \mathbb{R}, \tag{4.4}$$

$$\int_{\mathbb{R}^N} F(X,u(X))\mathrm{d}X \leq 0, \qquad \text{for all } u(X) \in S_k^2(\mathbb{R}^N), \tag{4.5}$$

where ρ, C_1, C_2 are positive constants, and $g(X) \in L^{N_k}(\mathbb{R}^N) \cap L^{\frac{N_k}{2}}(\mathbb{R}^N)$, $g_1(X) \in L^1(\mathbb{R}^N), F(X,\xi) = \int_0^\xi f(X,\tau)d\tau$.

We put

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \mathbb{A} = \begin{pmatrix} 0 & Id \\ G_k - Id & 0 \end{pmatrix},$$

$$f^*(U)(X) = \begin{pmatrix} 0 \\ -\beta v(X) + f(X,u(X)) \end{pmatrix}, U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

Then the problem (4.1) can be formulated as an abstract evolutionary equation

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t} = \mathbb{A}U + f^*(U), \quad t > 0, \\ U(0) = U_0. \end{cases} \tag{4.6}$$

We set $\mathcal{H} = S_k^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. We regard $\mathcal{H}$ as a Hilbert space with the inner product

$$(U, \overline{U})_{\mathcal{H}} = \left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} \right) = (u, \overline{u})_{S_k^2(\mathbb{R}^N)} + (v, \overline{v})_{L^2(\mathbb{R}^N)}.$$

4.1.2 Unique of mild solutions

Theorem 4.1.1. *Assume that $f(X, u)$ satisfies the conditions* (4.2)-(4.5) *and $U_0 \in \mathcal{H}$. Then the problem* (4.6) *has a unique global solution $U \in C([0, \infty); \mathcal{H})$. Moreover, for each fixed t the map $U_0 \to S(t)U_0 := U(t)$ is continuous on $\mathcal{H}$.*

4.2 Existence of a global attractor in $S_k^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$

In view of Theorem 4.1.1, we can define a continuous semigroup $S(t) : \mathcal{H} \to \mathcal{H}$ as follows

$$S(t)U_0 := U(t),$$

where $U(t)$ is the unique global mild solution of (4.6) subject to U_0 as the initial datum.

Lemma 4.2.1. *Assume that $f(X, u)$ satisfies the conditions* (4.2)-(4.5) *and $U_0 \in \mathcal{B}$. Then for every $\epsilon > 0$, there exist positive constants $T(\epsilon)$ and $K(\epsilon)$ such that the solution $U(t)$ of the problem* (4.6) *satisfies*

$$\int\limits_{|X|_k \geq R} \Big(|u|^2 + |u_t|^2 + |\nabla_k u|^2 \Big) \mathrm{d}X \leq \epsilon, \quad t \geq T(\epsilon), \quad R \geq R(\epsilon), \tag{4.7}$$

where

$$|X|_k = \Big[|x|^{2(1+k)} + (1+k)^2 |y|^2 \Big]^{\frac{1}{2(1+k)}}.$$

Theorem 4.2.2. *Assume that $f(X,u)$ satisfies the conditions* (4.2)-(4.5). *Then the semigroup $\{S(t)\}_{t\geq 0}$ associated with the problem* (4.6) *is asymptotically compact in the phase space $\mathcal{H}$, i.e., let $\{U_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathcal{H}$ and a time sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \to +\infty$ as $n \to +\infty$, then $\{S(t_n)U_n\}_{n=1}^{\infty}$ is precompact in $\mathcal{H}$.*

The result on the existence of global attracctors is presented in the following theorem.

Theorem 4.2.3. *Assume that $f(X,u)$ satisfies the conditions* (4.2)-(4.5). *Then, the problem* (4.1) *possesses a global attractor in $\mathcal{H}$ which is a compact invariant subset that attracts every bounded set of $\mathcal{H}$ with respect to the norm topology.*

CONCLUSION

1. RESULTS OF THE THESIS

In this thesis, we study the existence of solutions, the regularity of weak solution of boundary value problems containing degenerate elliptic equations in bounded domains with smooth boundary, global solutions, global attractors of the initial boundary value problem for a damped hyperbolic equation containing degenerate elliptic operators. The main results obtained in the thesis include

1. For semilinear degenerate elliptic boundary value problem: We prove the existence of weak solution of the problem under some conditions on nonlinear terms. We also prove the regularity of weak solutions of the problems.

2. For damped hyperbolic equations containing strongly degenerate elliptic operators in bounded domain: We prove the existence and uniqueness of mild solutions, the existence of compact connected global attractor in $S^2_{(k_1,k_2),0}(\Omega)\times L^2(\Omega)$ and establish the finite fractal dimensionality of the attractor.

3. For damped hyperbolic equations containing the Grushin operator in the whole space: We prove the existence and uniqueness of mild solutions and the existence of a global attractor in the spaces $S^2_k(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$.

4. Giving an example to illustrate the existence of weak solutions and the global attractors in the corresponding space.

2. OUTLOOK

Beside the results obtained in this thesis, some open issues need to be investigated.

1. Conditions on the existence of solution to the above boundary value problem in unbounded domains, in bounded domains with inhomogeneous Dirichlet boundary conditions.

2. The existence of attractors for time dependent nonlinear function such as uniform attractors, pullback attractors and properties of attractors.

3. The existence of solutions and asymptotic behaviour of solutions of degenerate hyperbolic equations with different boundary conditions such as inhomogeneous boundary conditions, Neumann boundary conditions, mixed boundary conditions, nonlinear boundary conditions, ...To do this, we need to build corresponding weighted spaces, embedding theorems of Sobolev's type.

4. Build models of the obtained results for application in practice.

THE AUTHOR'S PUBLICATION RELATED TO THE THESIS

1. D. T. Luyen and N. M. Tri (2012), On boundary value problem for semilinear degenerate elliptic differential equations, *AIP Conf. Proc.* 1450, 13–17.

2. D. T. Luyen and N. M. Tri (2015), Existence of solutions to boundary value problems for semilinear Δ_γ differential equations, *Mathematical Notes* Vol. 97,No. 1, 73–84.

3. D. T. Luyen and N. M. Tri (2016), Large - time behavior of solutions to damped hyperbolic equation involving strongly degenerate elliptic differential operators, *Siberian Mathematical Journal,* Vol. 57, No. 4, 632–649.

4. D. T. Luyen and N. M. Tri (2016), Global attractor of the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator, *Annales Polonici Mathematici* Vol. 117, 141–161.

The results of the thesis have been presented at:

1. Seminar of Department of Analysis, Faculty of Mathematics, Hanoi National University of Eduacation;

2. Seminar of Department of Differential Equations, Institute of Mathematics, VAST;

3. The 5th international conference on research and education in mathematics, Indonesia, 2012.